# Poisson processes and Upper Limits

Vinay Kashyap
CHASC AstroStatistics Collaboration
Smithsonian Astrophysical Observatory

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# Outline

I. Poisson likelihood

II. Intro to Bayesian Analysis

1. Bayes' Theorem
2. Priors
3. Credible Ranges

III. Aperture Photometry

IV. Upper Limits

# Outline

I. Poisson likelihood

II. Intro to Bayesian Analysis

1. Bayes' Theorem
2. Priors
3. Credible Ranges (also Confidence Intervals)

III. Aperture Photometry

IV. Upper Limits

# I. Poisson likelihood

Consider $N$ counts uniformly distributed over an interval $\tau$

Constant rate $R = N/\tau$

What is the probability of finding k counts in $\delta t$?

# I. Poisson likelihood

consider a randomly selected interval $\delta t$

$$\rho = \delta t/\tau \equiv R\delta t/N$$

finding $k$ counts in that interval (given there are $N$ total)

$$^{N}C_{k}\, \rho^{k}\, (1\text{-}\rho)^{N\text{-}k}$$

# I. Poisson likelihood

consider a randomly selected interval $\delta t$

$$\rho = \delta t/\tau \equiv R\delta t/N$$

finding $k$ counts in that interval (given there are $N$ total)

$${}^{N}C_{k}\, \rho^{k}\, (1\text{-}\rho)^{N\text{-}k}$$

$$p(k|R\,\delta t) = \frac{(R\,\delta t)^k}{k!} e^{-R\,\delta t}$$

# I. Poisson likelihood

finding $k$ counts in that interval (given there are $N$ total)

$$^{N}C_{k}\, \rho^{k}\, (1-\rho)^{N-k}$$

$$p(k|R\,\delta t) = \frac{(R\,\delta t)^k}{k!} e^{-R\,\delta t}$$

# I. Poisson likelihood

finding $k$ counts in that interval (given there are $N$ total)

$$^{N}C_k \, \rho^k \, (1-\rho)^{N-k}$$

$$\frac{N!}{(N-k)!k!}\left(\frac{R\,\delta t}{N}\right)^k\left(1-\frac{R\,\delta t}{N}\right)^{N-k}$$

$$p(k|R\,\delta t) = \frac{(R\,\delta t)^k}{k!}e^{-R\,\delta t}$$

# I. Poisson likelihood

finding $k$ counts in that interval (given there are $N$ total)

$$^{N}C_k \, \rho^k \, (1-\rho)^{N-k}$$

$$\frac{N!}{(N-k)!k!}\left(\frac{R\,\delta t}{N}\right)^k \left(1-\frac{R\,\delta t}{N}\right)^{N-k}$$

$$\frac{N!}{(N-k)!N^k}\,\frac{(R\,\delta t)^k}{k!}\left(1-\frac{R\,\delta t}{N}\right)^N \left(1-\frac{R\,\delta t}{N}\right)^{-k}$$

$$p(k|R\,\delta t) = \frac{(R\,\delta t)^k}{k!} e^{-R\,\delta t}$$

# I. Poisson likelihood

finding $k$ counts in that interval (given there are $N$ total)

$${}^{N}C_k\, \rho^k\, (1\text{-}\rho)^{N\text{-}k}$$

$$\frac{N!}{(N-k)!k!}\left(\frac{R\,\delta t}{N}\right)^k\left(1-\frac{R\,\delta t}{N}\right)^{N-k}$$

$$\frac{N!}{(N-k)!N^k}\,\frac{(R\,\delta t)^k}{k!}\left(1-\frac{R\,\delta t}{N}\right)^N\left(1-\frac{R\,\delta t}{N}\right)^{-k}$$

$$N\to\infty, \delta t\to 0 : \frac{N!}{(N-k)!N^k}\to 1, \left(1-\frac{R\,\delta t}{N}\right)^N\to e^{-R\,\delta t}$$

$$p(k|R\,\delta t)=\frac{(R\,\delta t)^k}{k!}e^{-R\,\delta t}$$

# II. Bayesian Analysis

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a calculus for ***conditional*** probabilities

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## a calculus for ***conditional*** probabilities

Notation

- $p(..)$
- $p(\mathrm{A})$ – probability of a proposition
- $p(\mathrm{AB})$ – probability of A *and* B
- $p(\mathrm{A|B})$ – probability of A *given* B
- $p(x)\mathrm{d}x$ – probability density (without the $\mathrm{d}x$)

All you need to remember

- $p(\mathrm{A}\ \textit{or}\ \mathrm{B}) = p(\mathrm{A}) + \mathrm{p(B)} - p(\mathrm{A}\ \textit{and}\ \mathrm{B})$
- $p(\mathrm{A}\ \textit{and}\ \mathrm{B}) = p(\mathrm{A}\ \textit{given}\ \mathrm{B}) \cdot p(\mathrm{B})$

# II.a Bayes' Theorem

$$p(AB) = p(A|B) \cdot p(B) = p(B|A) \cdot p(A)$$

$$p(A|B) = \frac{p(B|A) \cdot p(A)}{p(B)}$$

# II.a Bayes' Theorem

$$p(\theta|D) = \frac{p(D|\theta) \cdot p(\theta)}{p(D)}$$

# II.a Bayes' Theorem

likelihood

model parameters

Data

$$p(\theta|D) = \frac{p(D|\theta) \cdot p(\theta)}{p(D)}$$

prior

posterior probability

normalization

# II.b Priors

"Extraordinary claims require extraordinary evidence."

- Carl Sagan

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Why?

# II.b Priors

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- Carl Sagan

Why?

Because priors.

$$p(\theta|D) = \frac{p(D|\theta) \cdot p(\theta)}{p(D)}$$

# II.b Priors

- Unfairly maligned as “subjective”, but actually a mechanism to explicitly encode your assumptions
- When your data are weak, your prior beliefs don’t change; when your data are strong, your prior beliefs don’t matter.
- You update your prior belief with new data, using Bayes’ Theorem. Lets you daisy-chain analyses.
- When your prior is informative, takes more data to make a large change.
- Technically, the biggest difference between likelihood analysis and Bayesian analysis: converts $p(D|\theta)$ to $p(\theta|D)$

$$\frac{(R\,\delta t)^k}{k!} e^{-R\,\delta t}$$

# II.b Example: $\gamma$-Priors

- Highly flexible distribution, defined on non-negative reals, $[0,\infty)$
- Conjugate prior to the Poisson distribution

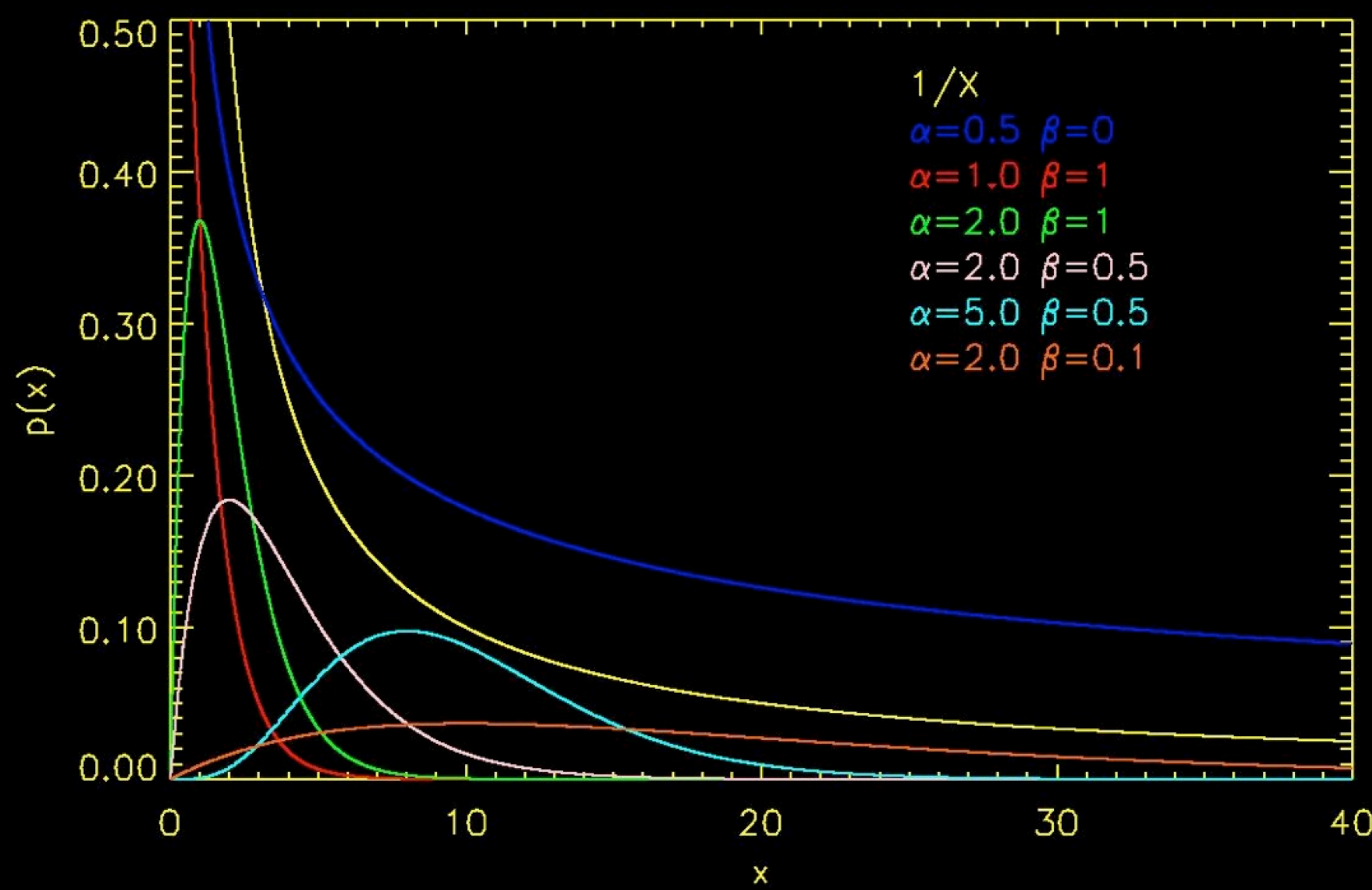
$$\gamma(x;\alpha,\beta) = x^{\alpha\text{-}1}\, e^{-\beta x} \cdot \beta^{\alpha} / \Gamma(\alpha)$$

- $\alpha$ = mean$^2$/variance, $\beta$ = mean/variance
- As $\alpha \rightarrow 1, \beta \rightarrow 0$, approaches a flat, non-informative prior
- For non-trivial $\alpha,\beta,$ acts as an informative prior where you expect to observe $\alpha$ counts in $\beta$ “exposure”

$$\gamma(x;\alpha,\beta) = x^{\alpha-1}\, e^{-\beta x} \cdot \beta^{\alpha} / \Gamma(\alpha)$$

$$mean = \alpha/\beta \qquad variance = \alpha/\beta^2$$

# II.b Example: γ-Priors



# II.c Confidence Ranges

The uncertainty in a parameter is defined by the width of its probability distribution.

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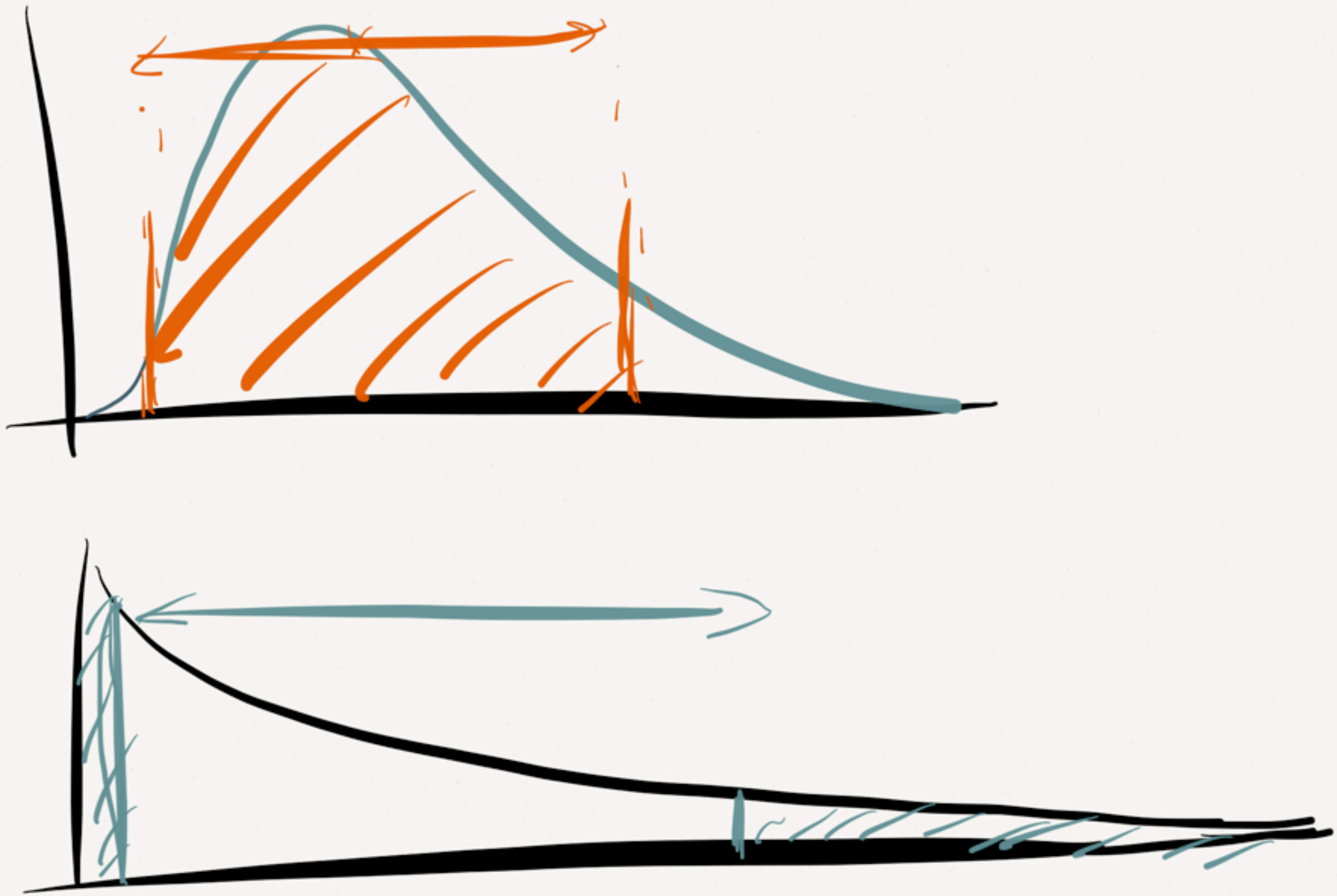
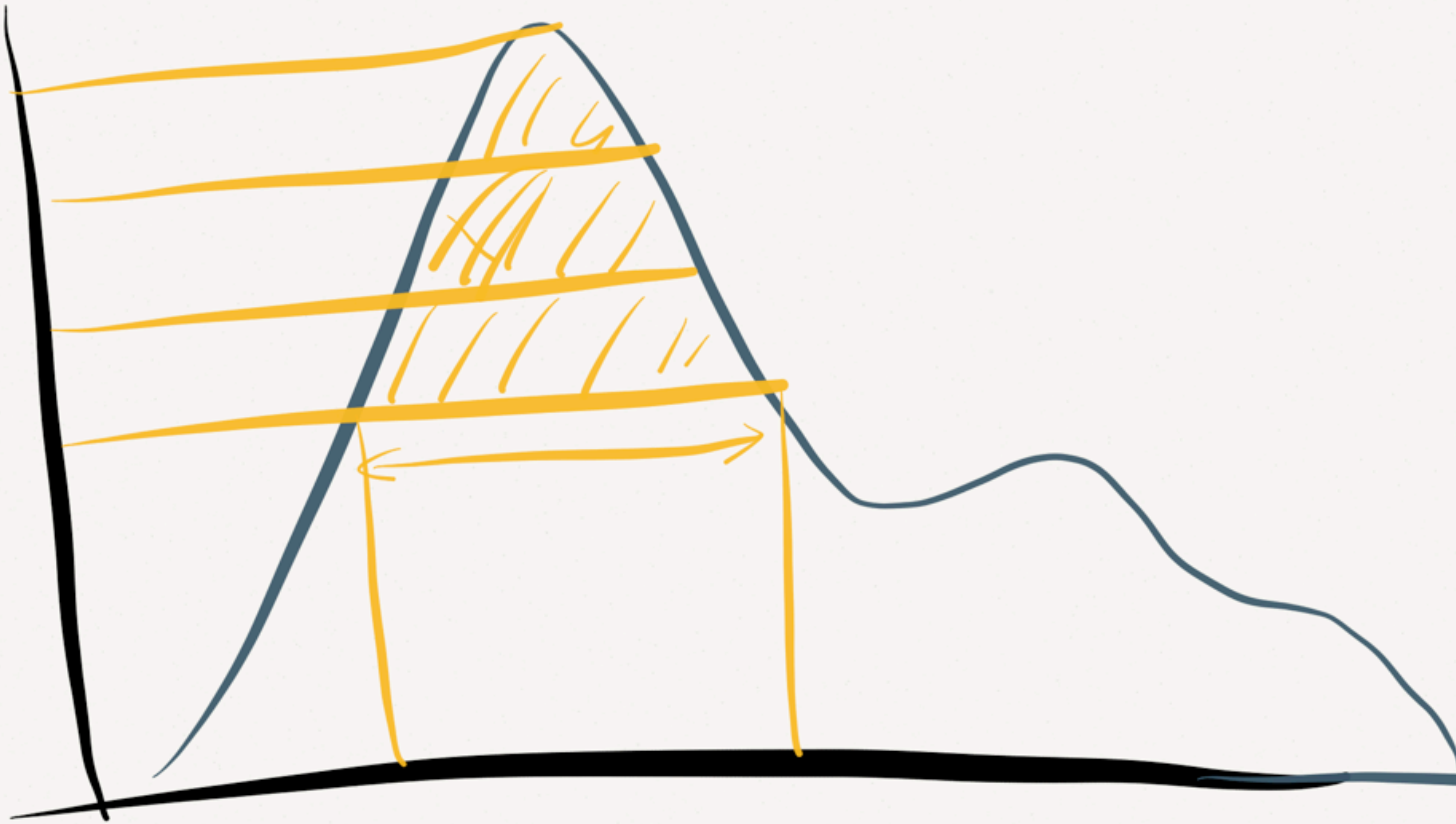
The uncertainty in a parameter is defined
by the width of its probability distribution.

- Frequentist confidence interval:
  - Intervals computed at some significance $p$ will contain the true value a fraction $p$ of the times the experiment is repeated
- Bayesian credible range:
  - An interval at significance $p$ will contain the true value of the parameter with probability $p$

# II.c.1 Credible Ranges

- Not unique!
- Set bounds on parameters
- many types: Equal-tail, Highest Posterior-density, Gaussian-equivalent $\sigma$, mode-outward, etc.





# II.c.1 Credible Ranges

- Not unique!
- Set bounds on parameters
- many types: Equal-tail, Highest Posterior-density, Gaussian-equivalent $\sigma$, mode-outward, etc.
  - Equal-tail is transformation invariant
  - HPD guaranteed to include mode; also smallest
  - Using Gaussian-equivalent $\pm\sigma$ is often a very bad idea

# II.c.2 Confidence Interval

Invert a hypothesis test: ask what is the likelihood of the data for different possible parameter values, and define a confidence region at level $1\text{-}\gamma$ as that set of parameters which are not rejected at significance $\gamma$.

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Invert a hypothesis test: ask what is the likelihood of the data for different possible parameter values, and define a confidence region at level $1\text{-}\gamma$ as that set of parameters which are not rejected at significance $\gamma$.

- e.g., Poisson when $n$ counts are observed:
  - upper bound, $s=s_u$ such that

$$1\text{-}\gamma = p(k \le n;\, s) = \sum_{k=0..n} s^k e^{-s} / \Gamma(k+1)$$

  - lower bound, $s=s_l$ such that

$$1\text{-}\gamma = p(k > n;\, s) = 1 - \sum_{k=0..n-1} s^k e^{-s} / \Gamma(k+1)$$

# II.c.2 Confidence Interval

Invert a hypothesis test: ask what is the likelihood of the data for different possible parameter values, and define a confidence region at level $1-\gamma$ as that set of parameters which are not rejected at significance $\gamma$.

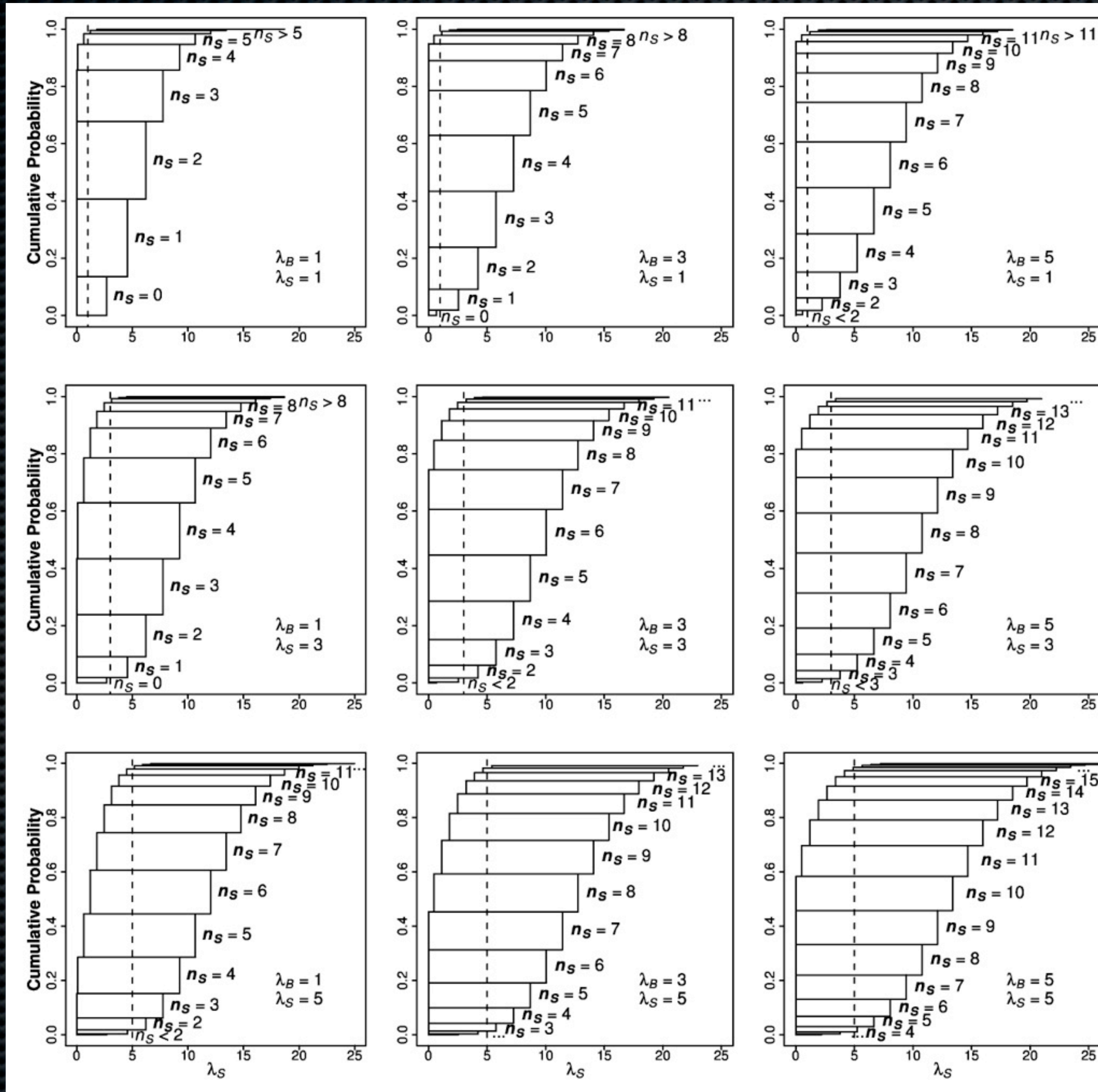
- e.g., Poisson ($n$ counts) with background ($b$, known)
  - upper bound, find $s=s_u$ (for given $b$) such that

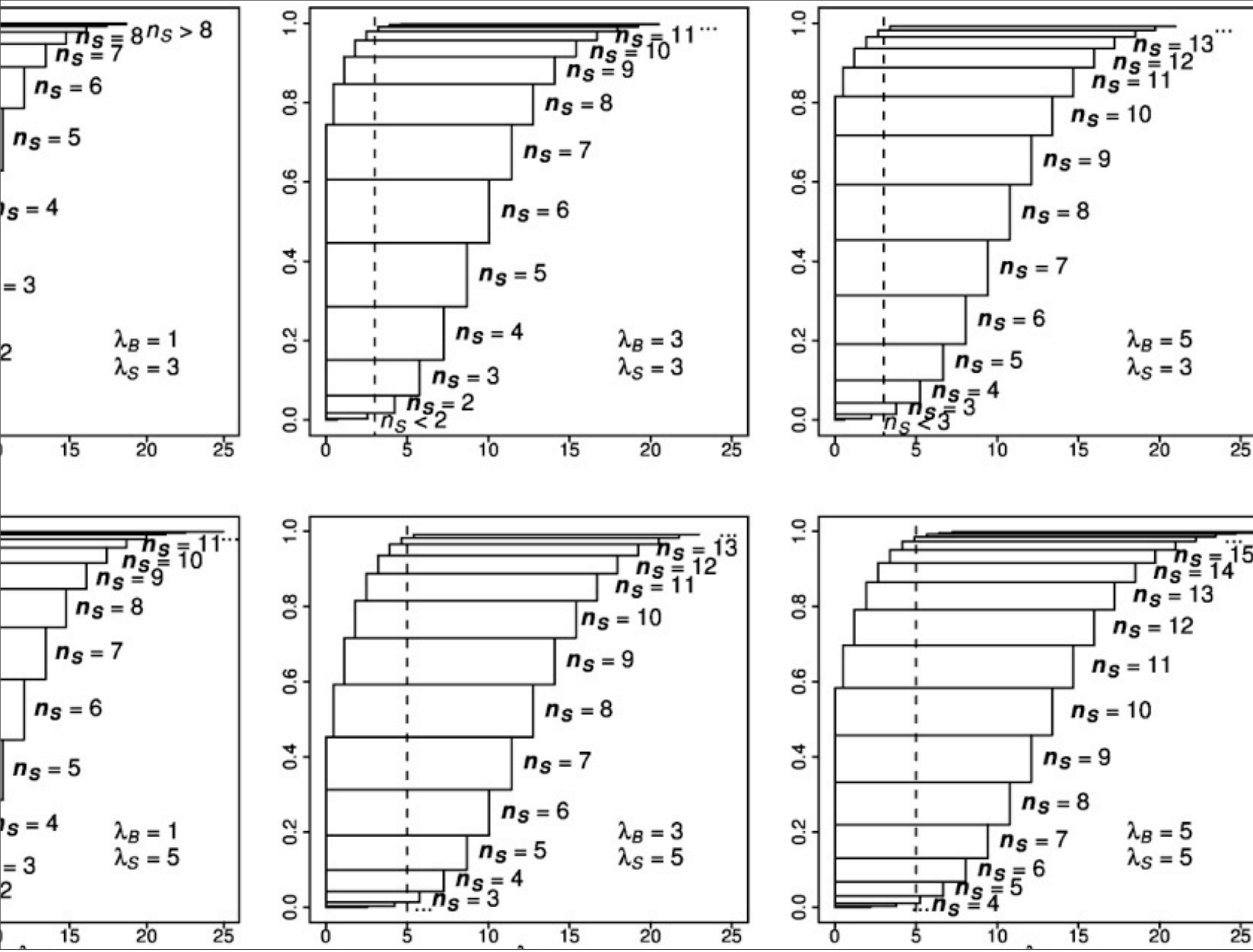
$$1-\gamma = p(k \leq n; s,b) = \sum_{k=0..n} (s+b)^k e^{-(s+b)} / \Gamma(k+1)$$

  - lower bound, find $s=s_l$ (for given $b$) such that

$$1-\gamma = p(k>n; s,b) = 1 - \sum_{k=0..n-1} (s+b)^k e^{-(s+b)} / \Gamma(k+1)$$

Confidence intervals for given background ($\lambda_B$) and when different counts ($n_S$) are observed. Width of boxes are 95% intervals. Height of boxes are $p(n_S|\lambda_S\lambda_B)$. Dashed vertical line is true value of $\lambda_S$.





$n_S = 8$ $n_S > 8$
$n_S = 7$
$n_S = 6$
$n_S = 5$
$n_S = 4$
$= 3$
$\lambda_B = 1$
$\lambda_S = 3$
$n_S = 11$ $n_S = 10$ $n_S = 9$ $n_S = 8$ $n_S = 7$ $n_S = 6$ $n_S = 5$ $n_S = 4$ $n_S = 3$ $n_S = 2$ $n_S < 2$
$\lambda_B = 3$
$\lambda_S = 3$
$n_S = 13$ $n_S = 12$ $n_S = 11$ $n_S = 10$ $n_S = 9$ $n_S = 8$ $n_S = 7$ $n_S = 6$ $n_S = 5$ $n_S = 4$ $n_S = 3$ $n_S < 3$
$\lambda_B = 5$
$\lambda_S = 3$
$n_S = 11$ $n_S = 10$ $n_S = 9$ $n_S = 8$ $n_S = 7$ $n_S = 6$ $n_S = 5$ $n_S = 4$ $= 3$
$\lambda_B = 1$
$\lambda_S = 5$
$n_S = 13$ $n_S = 12$ $n_S = 11$ $n_S = 10$ $n_S = 9$ $n_S = 8$ $n_S = 7$ $n_S = 6$ $n_S = 5$ $n_S = 4$ $n_S = 3$
$\lambda_B = 3$
$\lambda_S = 5$
$n_S = 15$ $n_S = 14$ $n_S = 13$ $n_S = 12$ $n_S = 11$ $n_S = 10$ $n_S = 9$ $n_S = 8$ $n_S = 7$ $n_S = 6$ $n_S = 5$ $n_S = 4$
$\lambda_B = 5$
$\lambda_S = 5$
0.0 0.2 0.4 0.6 0.8 1.0
0 5 10 15 20 25

# II.c.3 Feldman-Cousins Confidence Interval

Invert a hypothesis test: ask what is the likelihood of the data for different possible parameter values, and define a confidence region at level $1\text{-}\gamma$ as that set of parameters which are not rejected at significance $\gamma$.

- But: sometimes intervals can be empty (e.g., if $n \ll b$)
- invert the ratio of likelihoods,

$$l(s) = L(n|s,b) \,/\, L(n|\hat{s},b)$$

- unique, unified intervals where the lower bound automatically drops to $0$ for small $n$ – no need to select between one-sided and two-sided intervals

# III. Aperture Photometry

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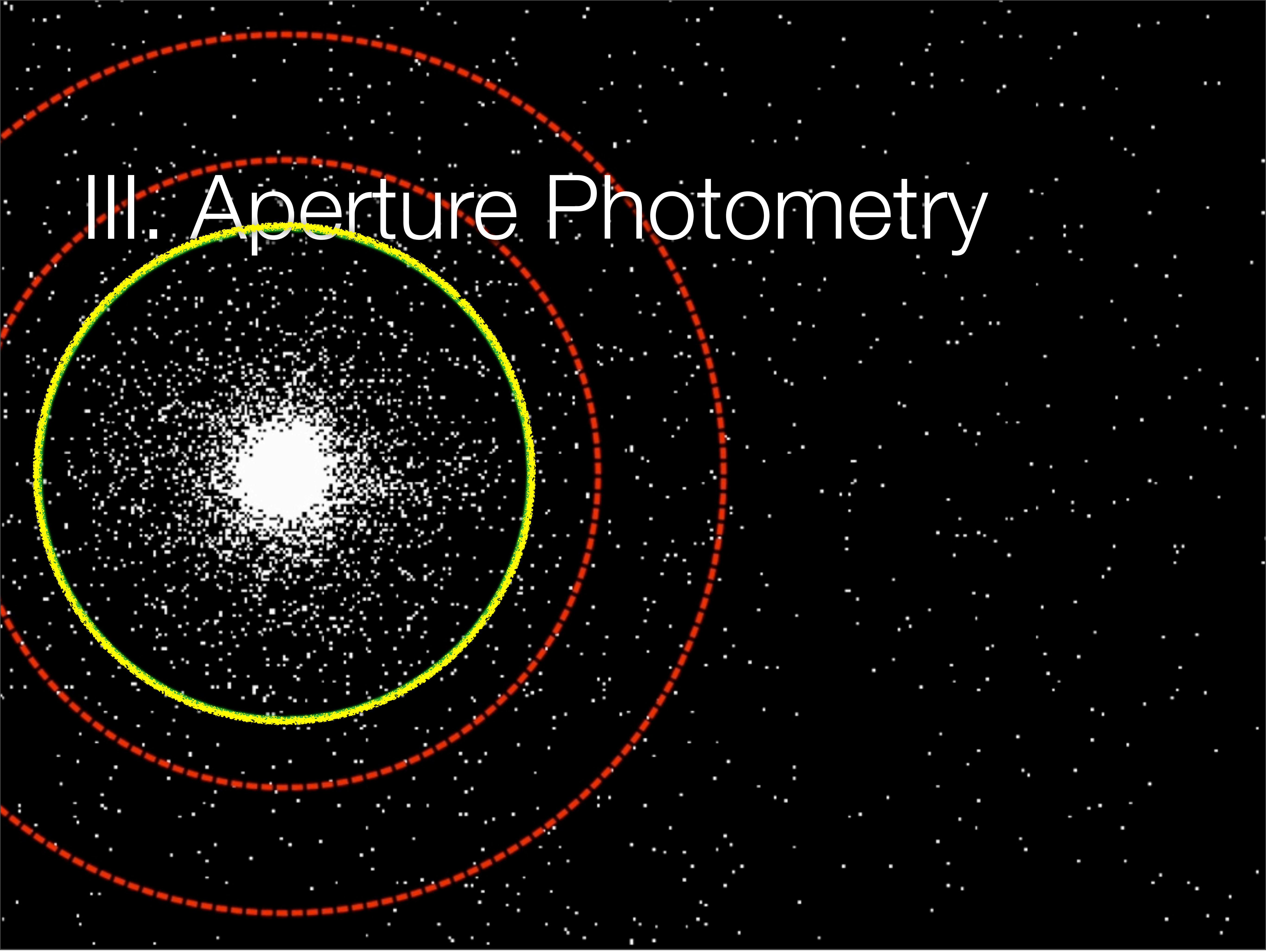
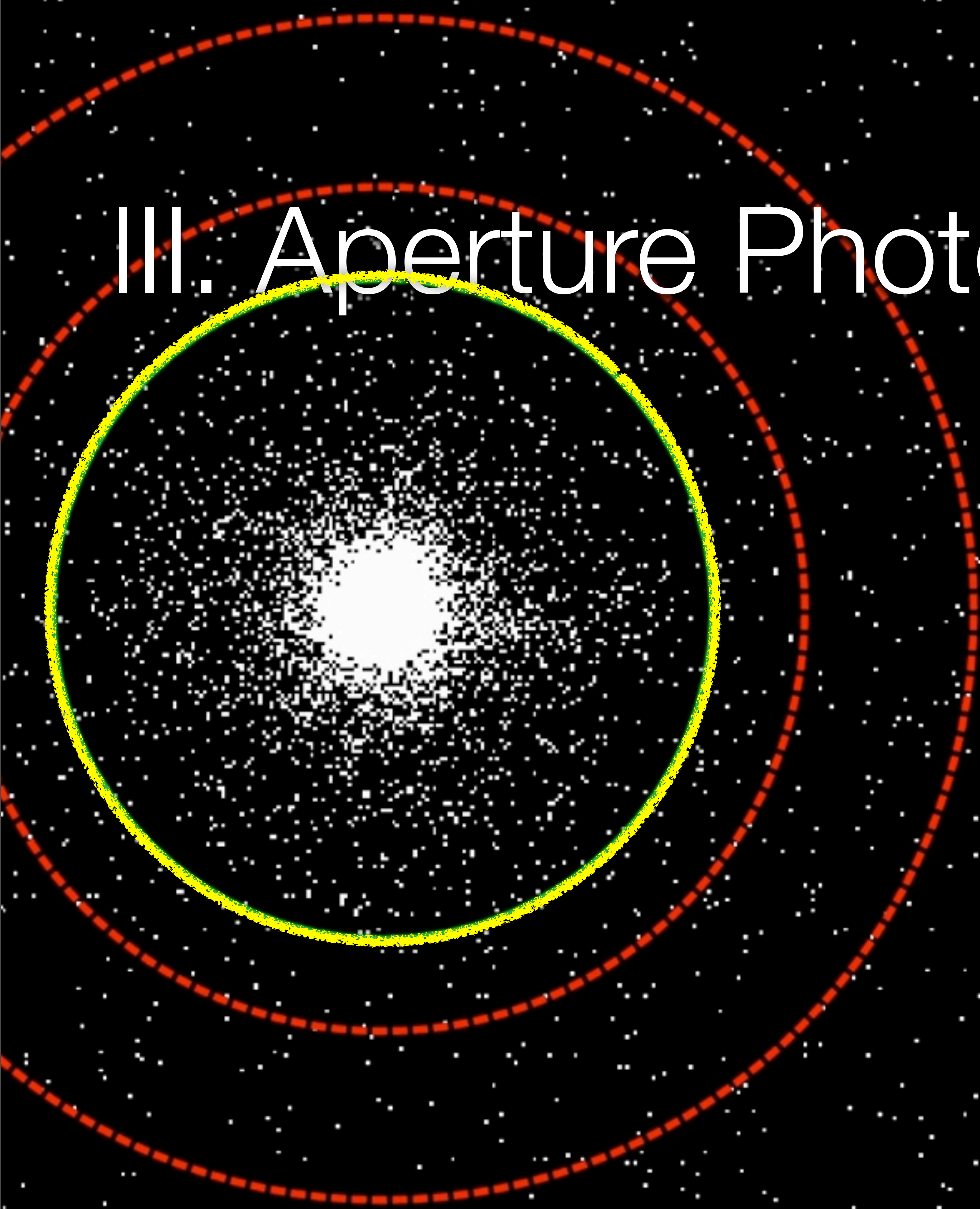


# III. Aperture Photometry



Given measured counts

**C** **B**

Infer expected counts

$\theta_S$ , $\theta_B$

**C** ~ $Pois(\theta_S + \theta_B)$

**B** ~ $Pois( r\ \theta_B )$

$$p(k|R\,\delta t) = \frac{(R\,\delta t)^k}{k!} e^{-R\,\delta t}$$

$$p(\theta|D) = \frac{p(D|\theta) \cdot p(\theta)}{p(D)}$$

$$\gamma(x;\alpha,\beta) = x^{\alpha-1}\, e^{-\beta x} \cdot \beta^{\alpha} / \Gamma(\alpha)$$

$$C \sim Pois(\theta_S + \theta_B)$$
$$B \sim Pois(\, r\, \theta_B\,)$$

# III. Aperture Photometry

$$p(k|R\,\delta t) = \frac{(R\,\delta t)^k}{k!} e^{-R\,\delta t}$$

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$$C \sim Pois(\theta_S + \theta_B)$$
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# III. Aperture Photometry

$$p(\theta_S) = \theta_S{}^{\alpha_S - 1}\, e^{-\beta_S\, \theta_S}\, \beta_S{}^{\alpha_S} / \Gamma(\alpha_S)$$

$$p(\theta_B) = \theta_B{}^{\alpha_B - 1}\, e^{-\beta_B\, \theta_B}\, \beta_B{}^{\alpha_B} / \Gamma(\alpha_B)$$

$$p(k|R\,\delta t) = \frac{(R\,\delta t)^k}{k!} e^{-R\,\delta t}$$

$$p(\theta|D) = \frac{p(D|\theta) \cdot p(\theta)}{p(D)}$$

$$\gamma(x;\alpha,\beta) = x^{\alpha\text{-}1}\ e^{-\beta x} \cdot \beta^{\alpha} / \Gamma(\alpha)$$

$$C \sim Pois(\theta_S + \theta_B)$$
$$B \sim Pois(\ r\ \theta_B\ )$$

# III. Aperture Photometry

$$p(\theta_S) = \theta_S^{\alpha_S\text{-}1}\ e^{-\beta_S\ \theta_S}\ \beta_S^{\alpha_S} / \Gamma(\alpha_S)$$

$$p(\theta_B) = \theta_B^{\alpha_B\text{-}1}\ e^{-\beta_B\ \theta_B}\ \beta_{\mathbf{B}}^{\alpha_B} / \Gamma(\alpha_B)$$

$$p(B|\theta_B) = (r\theta_B)^B\ e^{-r\theta_B} / \Gamma(B+1)$$

$$p(C|\theta_S\theta_B) = (\theta_S+\theta_B)^C\ e^{-(\theta_S+\theta_B)} / \Gamma(C+1)$$

$$p(k|R\,\delta t) = \frac{(R\,\delta t)^k}{k!} e^{-R\,\delta t}$$

$$p(\theta|D) = \frac{p(D|\theta) \cdot p(\theta)}{p(D)}$$

$$\gamma(x;\alpha,\beta) = x^{\alpha-1}\, e^{-\beta x} \cdot \beta^{\alpha} / \Gamma(\alpha)$$

$$C \sim Pois(\theta_S + \theta_B)$$
$$B \sim Pois(\, r\, \theta_B\,)$$

# III. Aperture Photometry

$$p(\theta_S) = \theta_S^{\alpha_S-1}\, e^{-\beta_S\, \theta_S}\, \beta_S^{\alpha_S} / \Gamma(\alpha_S)$$

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$$p(B|\theta_B) = (r\theta_B)^B\, e^{-r\theta_B} / \Gamma(B+1)$$

$$p(C|\theta_S\theta_B) = (\theta_S+\theta_B)^C\, e^{-(\theta_S+\theta_B)} / \Gamma(C+1)$$

$$p(\theta_S\theta_B|C,B) \propto p(C,B|\theta_B\theta_S)\, p(\theta_S)\, p(\theta_B)$$

$$p(k|R\,\delta t) = \frac{(R\,\delta t)^k}{k!} e^{-R\,\delta t}$$

$$p(\theta|D) = \frac{p(D|\theta) \cdot p(\theta)}{p(D)}$$

$$\gamma(x;\alpha,\beta) = x^{\alpha\text{-}1}\ e^{-\beta x} \cdot \beta^{\alpha} / \Gamma(\alpha)$$

$$C \sim Pois(\theta_S + \theta_B)$$
$$B \sim Pois(\ r\ \theta_B\ )$$

# III. Aperture Photometry

$$p(\theta_S) = \theta_S{}^{\alpha_S\text{-}1}\ e^{-\beta_S\ \theta_S}\ \beta_S{}^{\alpha_S} / \Gamma(\alpha_S)$$

$$p(\theta_B) = \theta_B{}^{\alpha_B\text{-}1}\ e^{-\beta_B\ \theta_B}\ \beta_B{}^{\alpha_B} / \Gamma(\alpha_B)$$

$$p(B|\theta_B) = (r\theta_B)^B\ e^{-r\theta_B} / \Gamma(B+1)$$

$$p(C|\theta_S\theta_B) = (\theta_S+\theta_B)^C\ e^{-(\theta_S+\theta_B)} / \Gamma(C+1)$$

$$p(\theta_S\theta_B|C,B) \propto p(C,B|\theta_B\theta_S)\ p(\theta_S)\ p(\theta_B)$$

$$p(\theta_S\theta_B|C,B) \propto p(C|\theta_B\theta_S)\ p(B|\theta_B)\ p(\theta_S)\ p(\theta_B)$$

$$p(k|R\,\delta t) = \frac{(R\,\delta t)^k}{k!} e^{-R\,\delta t}$$

$$p(\theta|D) = \frac{p(D|\theta) \cdot p(\theta)}{p(D)}$$

$$\gamma(x;\alpha,\beta) = x^{\alpha-1}\, e^{-\beta x} \cdot \beta^{\alpha} / \Gamma(\alpha)$$

$$C \sim Pois(\theta_S + \theta_B)$$
$$B \sim Pois(r\, \theta_B)$$

# III. Aperture Photometry

$$p(\theta_S) = \theta_S^{\alpha_S-1}\, e^{-\beta_S\, \theta_S}\, \beta_S^{\alpha_S} / \Gamma(\alpha_S)$$

$$p(\theta_B) = \theta_B^{\alpha_B-1}\, e^{-\beta_B\, \theta_B}\, \beta_B^{\alpha_B} / \Gamma(\alpha_B)$$

$$p(B|\theta_B) = (r\theta_B)^B\, e^{-r\theta_B} / \Gamma(B+1)$$

$$p(C|\theta_S\theta_B) = (\theta_S+\theta_B)^C\, e^{-(\theta_S+\theta_B)} / \Gamma(C+1)$$

$$p(\theta_S\theta_B|C,B) \propto p(C,B|\theta_B\theta_S)\, p(\theta_S)\, p(\theta_B)$$

$$p(\theta_S\theta_B|C,B) \propto p(C|\theta_B\theta_S)\, p(B|\theta_B)\, p(\theta_S)\, p(\theta_B)$$

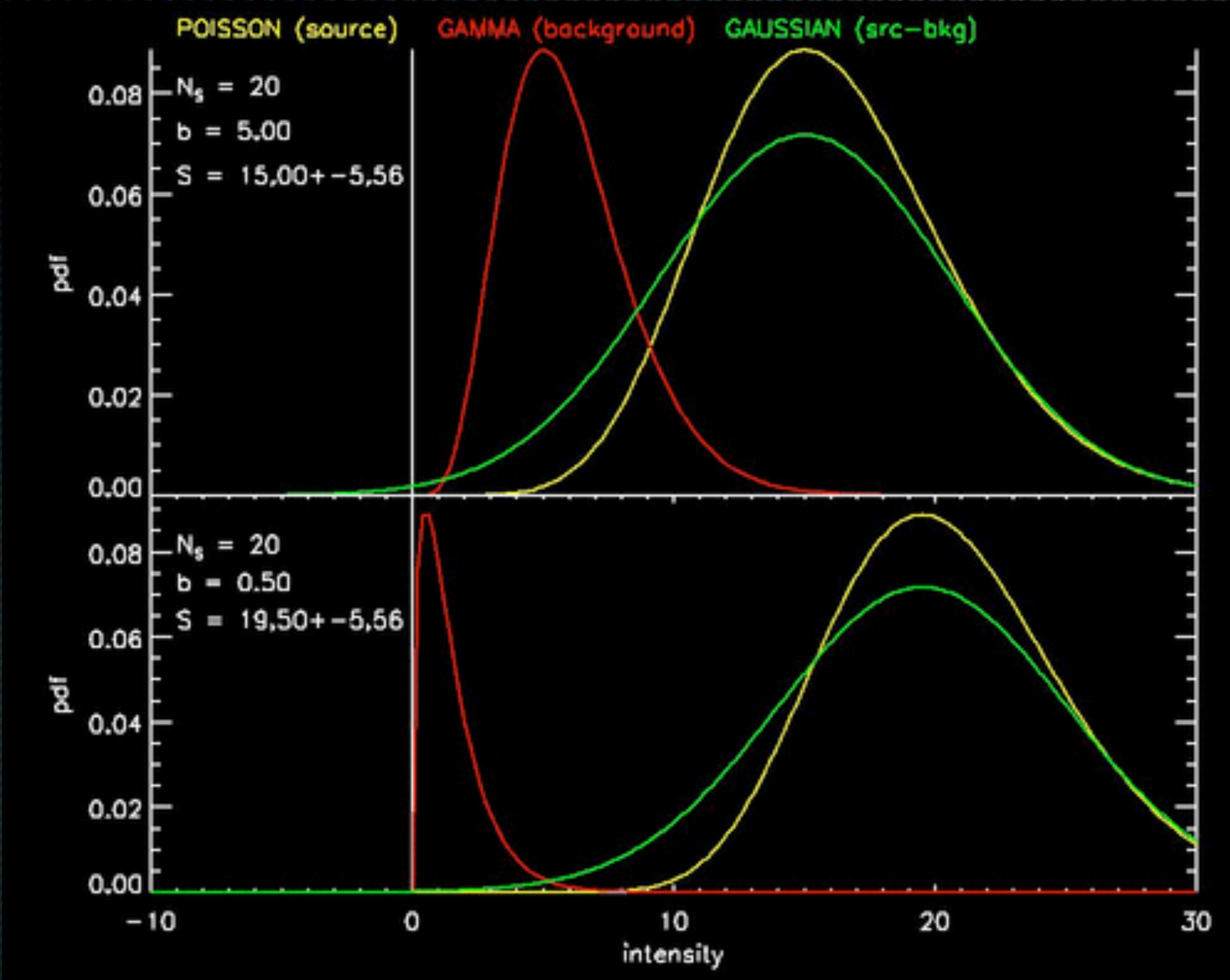
$$p(\theta_S|C,B) \propto \int d\theta_B\;\, p(C|\theta_B\theta_S)\, p(B|\theta_B)\, p(\theta_S)\, p(\theta_B)$$

# III. Aperture Photometry

$$p(\theta_S|C,B)\,d\theta_S = d\theta_S \frac{1}{\Gamma(C+1)\Gamma(B+1)}$$

$$\times \sum_{k=0}^{C} (r^{B+1} {\theta_S}^k e^{-\theta_S}$$

$$\times \frac{\Gamma(C+1)\Gamma(C+B-k+1)}{\Gamma(k+1)\Gamma(C-k+1)(1+r)^{C+B-k+1}})$$


POISSON (source)
GAMMA (background)
GAUSSIAN (src−bkg)
$N_s$ = 20
b = 5.00
S = 15.00+−5.56
$N_s$ = 20
b = 0.50
S = 19.50+−5.56
pdf
pdf
intensity
0.00
0.02
0.04
0.06
0.08
−10
0
10
20
30

# IV. Upper Limits

- A confidence interval or a credible range gives a range of values that a parameter can have for a specified significance.
- The interval has two ends.  A lower bound, and an upper bound.  The true value is likely higher than the lower bound.  And lower than the upper bound.
- Why is this not an *upper limit*?

# IV. Upper Limit

The largest intensity a source can have without being detected.

The smallest intensity a source should have to be detected.

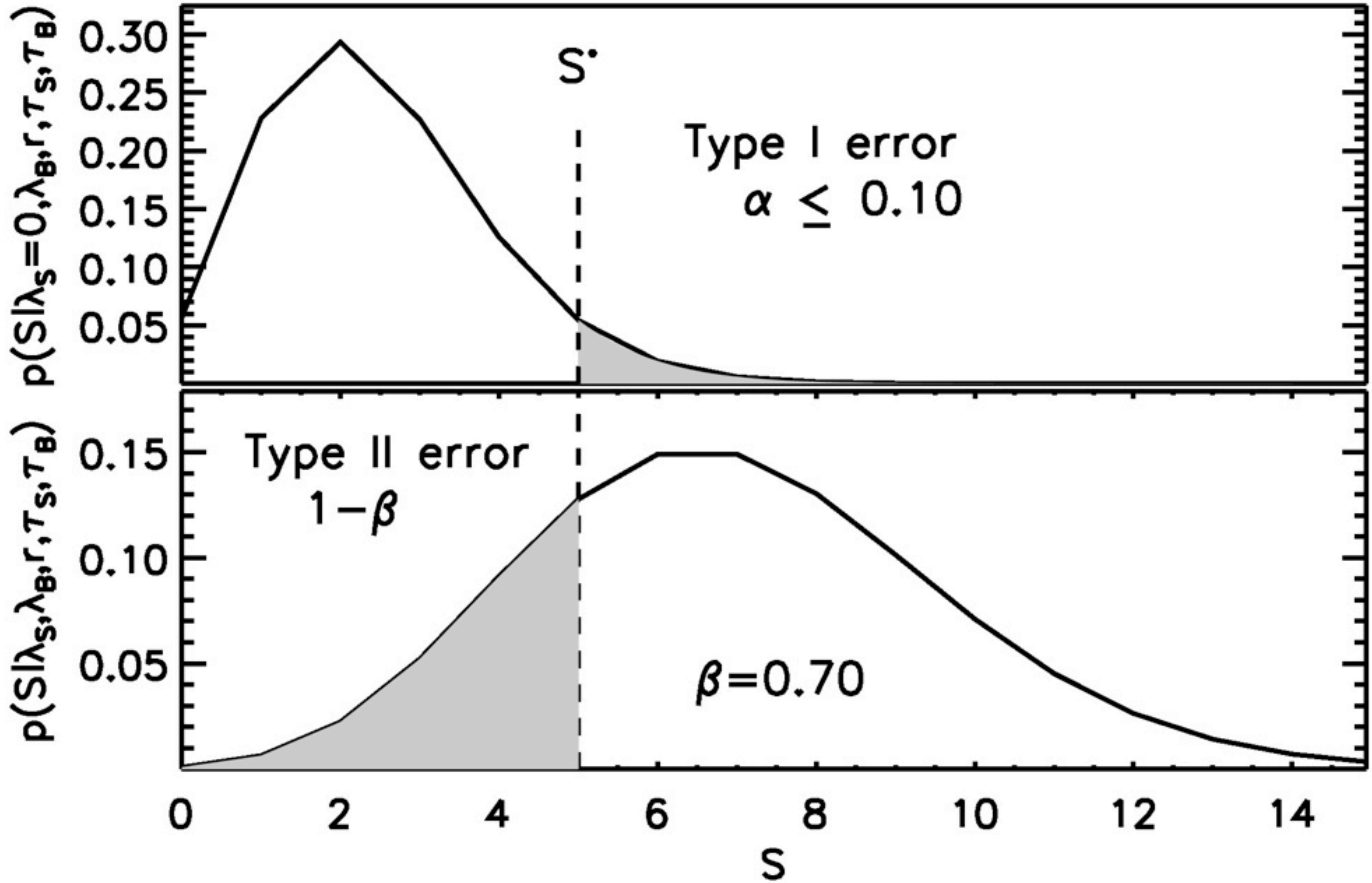
# IV. Upper Limit

- We define an upper limit in the context of *detection*
- Something is *detected* when some measurable statistic that is a function of the observed data exceeds a pre-set *threshold*
- e.g., test statistic $\mathfrak{S} \equiv n_S$ and threshold $\mathfrak{S}^* \equiv 5$ counts. If more than 5 counts are seen, claim detection. If fewer are seen, the source must be less bright than some value, aka *Upper Limit*
- Need both Type I and Type II errors to define Upper Limits

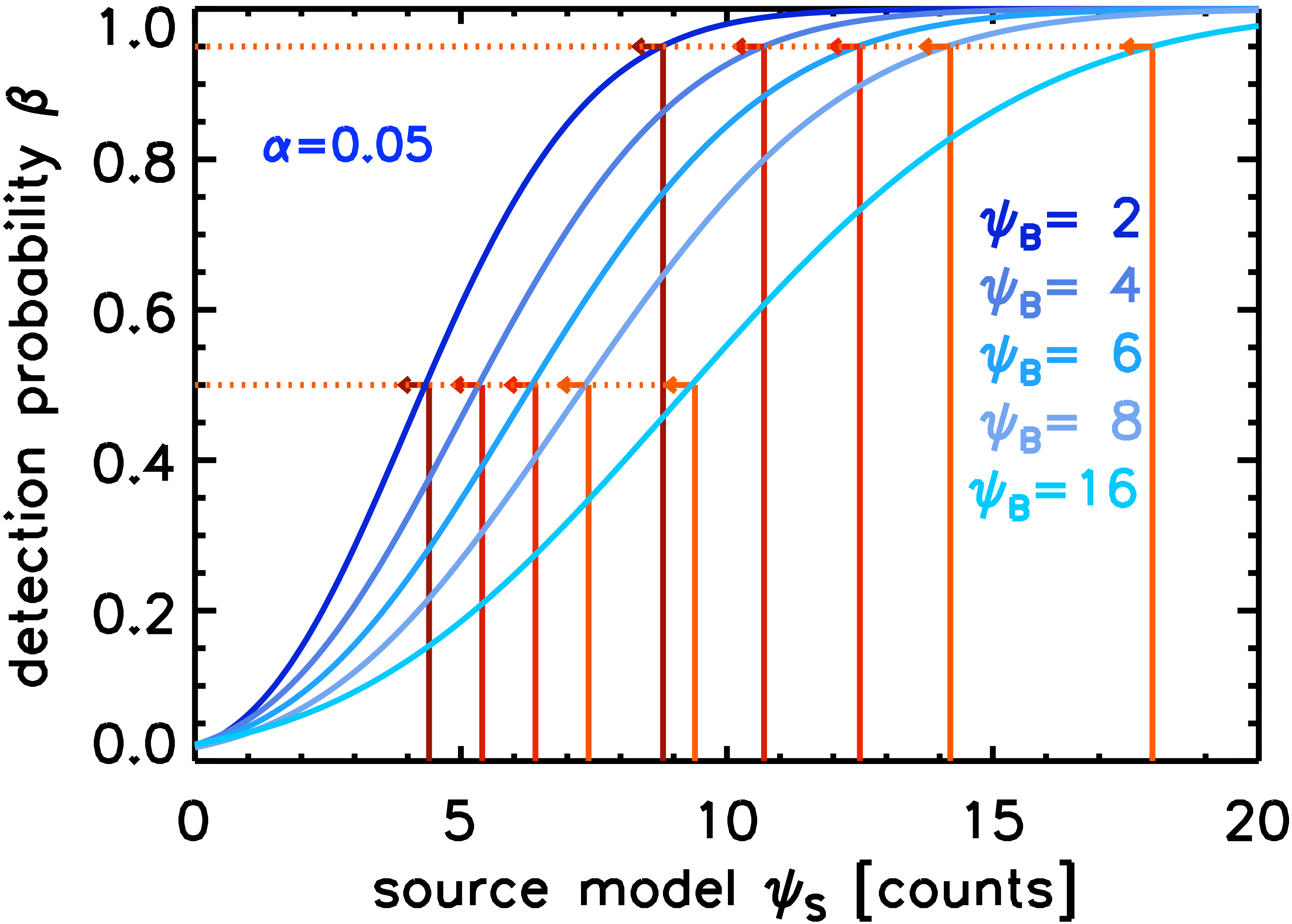
# IV. Upper Limit

- Suppose the threshold $\mathfrak{S}^*$ is defined by a false positive probability of $\alpha$ (e.g., the probability that a background fluctuation results in test statistic value $\mathfrak{S}>\mathfrak{S}^*$)
- A source with intensity $\theta_S$ will produce a signal that falls below the threshold $\mathfrak{S}^*$ with false negative probability $1\text{-}\beta$
- $U(\alpha,\beta)$ is the upper limit on $\theta_S$ such that

$$Pr(\mathfrak{S}>\mathfrak{S}^*(\alpha)|\theta_S,\theta_B) \geq \beta$$

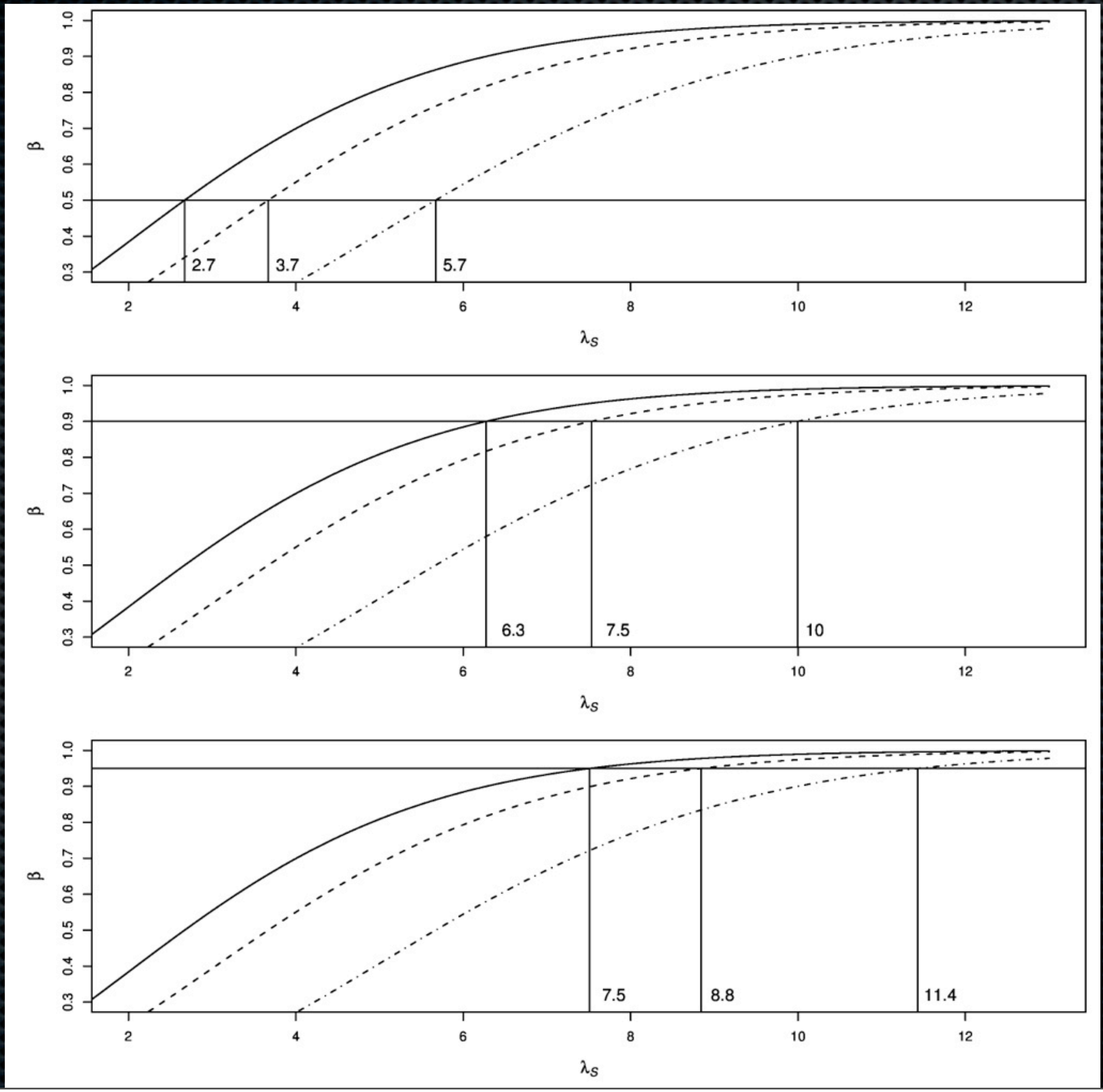


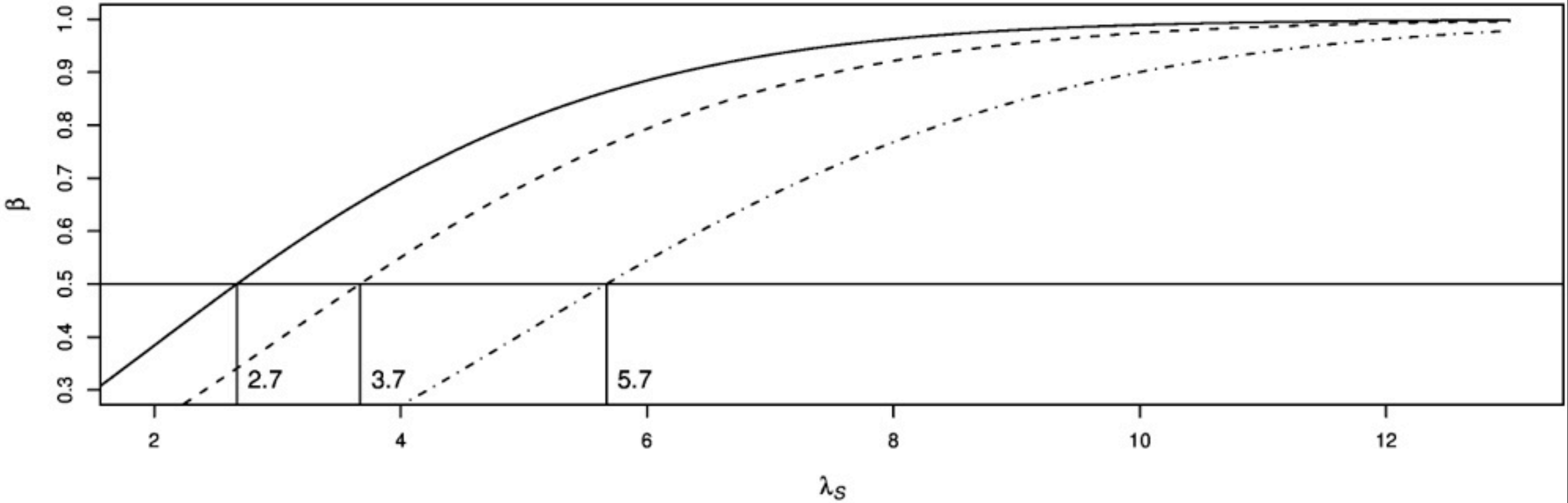
p(S|λ_S=0,λ_B,r,τ_S,τ_B)
0.30
0.25
0.20
0.15
0.10
0.05
S*
Type I error
α ≤ 0.10
p(S|λ_S,λ_B,r,τ_S,τ_B)
0.15
0.10
0.05
Type II error
1−β
β=0.70
0
2
4
6
8
10
12
14
S

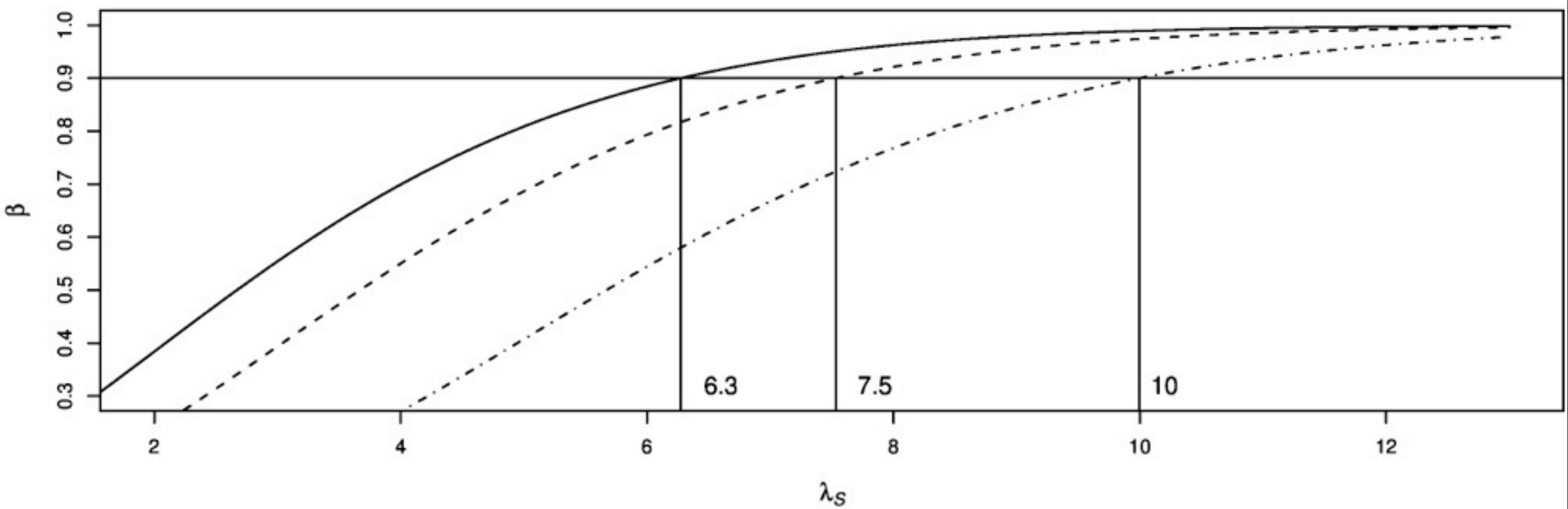


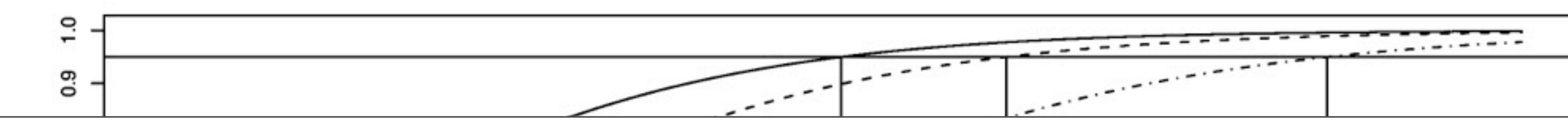
detection probability β
source model ψS [counts]
α=0.05
ψB= 2
ψB= 4
ψB= 6
ψB= 8
ψB=16
1.0
0.8
0.6
0.4
0.2
0.0
0
5
10
15
20

Upper limits for different choices of $\alpha$, $\mathfrak{S}^*$, and $\beta_{min}$, for a background of 5 counts in 10x source area. Curves are for $\mathfrak{S}^*=5, \alpha=0.1$ (solid), $\mathfrak{S}^*=6, \alpha=0.05$ (dashed), $\mathfrak{S}^*=8, \alpha=0.01$ (dash-dotted). Intercepts are for $\beta_{min}=0.5$ (top), $0.9$ (middle), $0.95$ (bottom).




β
λ_S
2.7
3.7
5.7
0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
2 4 6 8 10 12


β
λ_S
6.3
7.5
10
0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
2 4 6 8 10 12


0.9
1.0

# IV. Upper Limit – Properties

- Depends on the detection process (**`wavdetect`** will produce different upper limits than **`celldetect`**)
- Does *not* depend on the number of counts in source region
- *Does* depend on the background and exposure

# IV. Upper Limit – Recipe

1. Define a test statistic $\mathfrak{S}$ for measuring the strength of a source signal
2. Set the max probability of a false detection, $\alpha$ (e.g., $\alpha=0.003$ for a "$3\sigma$" detection) and compute the corresponding detection threshold $\mathfrak{S}^*(\alpha)$
3. Compute the probability of detection $\beta(\theta_S)$ for $\mathfrak{S}^*$
4. Define the min probability of detection $\beta_{min}$ (e.g., $\beta_{min}=0.5$)
5. Compute upper limit as value of $\theta_S$ such that $\beta(\theta_S) \geq \beta_{min}$.

# Further Reading

- Loredo 1990, Maximum Entropy and Bayesian Methods, Kluwer, Dordrecht, 81-142 : *Bayesian inference in Astrophysics*
  http://bayes.wustl.edu/gregory/articles.pdf
- Isobe, Feigelson, & Nelson, 1986, ApJ 306, 490 : *Statistical Methods for astronomical data wth upper limits*
- Feldman & Cousins, 1998, Phys. Rev. D, 57, 3873 : *Unified confidence intervals*
- van Dyk, Connors, Kashyap, & Siemiginowska, 2001, ApJ 548, 224 : *Bayesian analysis of spectra with low counts*
- Protassov, van Dyk, Connors, Kashyap, & Siemiginowska, 2002, ApJ, 571, 545 : *Limitations of Likelihood Ratio Tests*
- Park, Kashyap, Siemiginowska, van Dyk, Zezas, Heinke, & Wargelin, 2006, ApJ, 652, 610 : *Bayesian hardness ratios*
- Kashyap, van Dyk, Connors, Freeman, Siemiginowska, Xu, & Zezas, 2010, ApJ, 719, 900 : *Upper limits*
- Primini & Kashyap, 2014, circulated : *Aperture photometry for overlapping sources*